Non-commutative geometry and irreversibility

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Abstract. A kinetics built upon q-calculus, the calculus of discrete dilatations, is shown to describe diffusion on a hierarchical lattice. The only observable on this ultrametric space is the "quasi-position" whose eigenvalues are the levels of the hierarchy, corresponding to the volume of phase space available to the system at any given time. Motion along the lattice of quasi-positions is irreversible.

PACS. 05.20.Dd Kinetic theory – 05.70.Ln Nonequilibrium thermodynamics, irreversible processes

1 Introduction

The study of systems that are symmetric under dilatations rather than translations have been with us for a long time. The field of critical phenomena has been a breeding ground for useful scaling ideas. Fractal geometry [1] has provided us with a suitable language to describe systems with affine symmetries. It has been recently demonstrated [2] that, although ordinary derivatives of fractal or multifractal distributions may be nowhere defined, the finite q-derivative [3] provides a natural extension of the derivative to systems with discrete dilatation symmetries, and the q-integral [4] provides the requisite tool for integrating along a discrete path in scale space [5].

An appropriate language in which to describe the kinetics and dynamics of motion on such spaces, however, has not yet been sufficiently elaborated. One may think that a periodic lattice on the logarithmic scale would play here the same role as the linear chain does with respect to motion on a discrete, translationally invariant space. However, it is easy to see that progress on the logarithmic lattice is not symmetric with respect to a simultaneous reflection and time reversal, and corresponds to very different physics. It is the purpose of this paper to explore this asymmetry and to show that taking a statistical view point and associating the kinetics of a point on the logarithmic lattice with the motion of a representative point in phase space, leads naturally to the arrow of time one encounters in statistical physics [6].

The paper is organized as follows. In the next section, we will briefly recall the work of Dimakis and Müller-Hoissen [7] relating the so called q-deformed quantum mechanics [8] to quantum mechanics on a discrete lattice and then make a different, and we claim more natural, choice for the operators, to describe a different physics. In this

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description, the energy and momentum are no longer observables, nor are they conserved. Instead, we define a "quasi-position" operator, and show, in Section 3, that this tells us the volume in phase space over which the probability distribution of the representative point of our system is spread. In Section 4 we discuss connections with other recent work.

2 A Hamiltonian system on a hierarchical space

It has been demonstrated by Dimakis and Müller-Hoissen [7] that q-calculus [3,4] can be obtained from discrete calculus on a lattice by an exponential coordinate transformation. Under this transformation discrete translations go over to discrete dilatations. The q-deformed commutation relations obeyed by the transformed variables and their q-derivatives lead to q-deformed quantum mechanics [8]. In this way, q-deformed quantum mechanics has been given an interpretation in terms of quantum mechanics on a lattice.

Let us recall the definition of the q-derivative [3,7,9, 10]

$$\partial_y^{(q)} f(y) \equiv \frac{f(qy) - f(y)}{(q-1)y}, \qquad (1)$$

and

$$\overline{\partial}_y^{(q)} f(y) \equiv \frac{f(q^{-1}y) - f(y)}{(q^{-1} - 1)y}, \qquad (2)$$

where the subscript indicates the variable with respect to which the derivative is to be taken. It is easy to see that [7] these operators can be obtained from the discrete partial

derivatives

$$\tilde{\partial}_x^{(a)} f(x) \equiv \frac{1}{a}[f(x+a) - f(x)] \tag{3}$$

$$\overline{\tilde{\partial}}_x^{(a)} f(x) \equiv \frac{1}{a}[f(x) - f(x-a)] \tag{4}$$

under the coordinate transformation

$$y = q^{\frac{x}{q-1}} \tag{5}$$

$$q = 1 + a. \tag{6}$$

With $x = \ell a$, one has $y = q^\ell$, since $q - 1 = a$. Thus, this coordinate transformation takes the one-dimensional lattice with lattice spacing a, to another lattice which has spacing q on the logarithmic scale (Fig. 1a). It is useful to define discrete translation operators on the lattice in x-space, in terms of which the difference operator can be expressed; under the change of variables these go over to the discrete dilatation operators such that $(A_y^{(q)} f)(y) = f(qy)$ and $(\overline{A}_y^{(q)} f)(y) = f(q^{-1}y)$, with

$$A_y^{(q)} \equiv 1 + (q-1)Y\partial_y^{(q)} \tag{7}$$

and

$$\overline{A}_y^{(q)} \equiv 1 - (q-1)\frac{1}{q}Y\overline{\partial}_y^{(q)}. \tag{8}$$

If the position operator X is defined as multiplication by x, we notice that it is self-adjoint, and therefore can be identified with an observable, while the one sided (3, 4) difference operators $\tilde{\partial}_x^{(a)}$ and $\overline{\tilde{\partial}}_x^{(a)}$ are not. On the other hand, Dimakis and Müller-Hoissen [7] define the momentum and Hamiltonian operators in x-space via self-adjoint linear combinations of these operators, viz.,

$$P \equiv \frac{1}{2i}(\tilde{\partial}_x^{(a)} + \overline{\tilde{\partial}}_x^{(a)}) \tag{9}$$

and

$$H \equiv -\frac{1}{2}(\tilde{\partial}_x^{(a)}\overline{\tilde{\partial}}_x^{(a)}) = -\frac{1}{2a}(\tilde{\partial}_x^{(a)} - \overline{\tilde{\partial}}_x^{(a)}). \tag{10}$$

The momentum and Hamiltonian thus obtained satisfy the Heisenberg equations of motion; however, the usual canonical commutation relation is altered. Explicit calculation yields
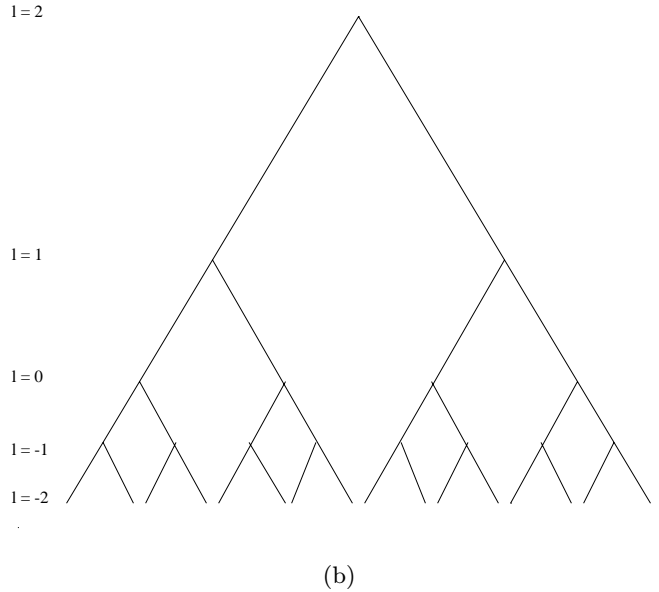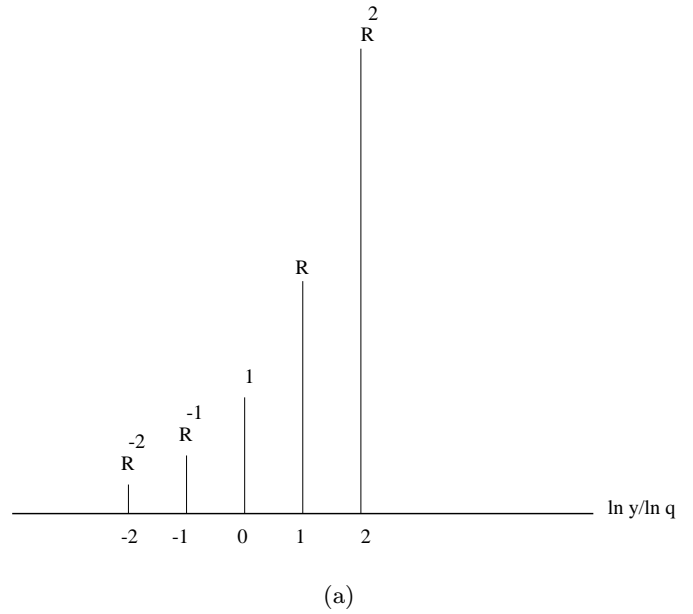
$$[P, X] = -i(1 - a^2 H). \tag{11}$$

In terms of the discrete time derivative, $\tilde{\partial}_t^{(\Delta t)} = (\tilde{T} - 1)/(\Delta t)$, where $\tilde{T}$ is the time translation operator by the finite increment Δt, the Schrödinger equation is written

$$\tilde{\partial}_t^{(\Delta t)} f(x, t) = H f(x, t). \tag{12}$$

This leads to the relation

$$\tilde{T} = 1 - iH\Delta t. \tag{13}$$

(a)

(b)

Fig. 1. The hierarchical space on which the quasi-position operator Y is defined. The "quantum number" ℓ corresponds to the highest node which the particle has so far surmounted, with an associated energy barrier of height R^ℓ, thereby being delocalized over a region of size $q^\ell y_0$. We will define the distance d between two states such that $q^d y_0$ is the smallest interval which contains both of them. Thus, the quasi-position ℓ is the upper bound on the distance d to the origin at any given time. Note that we allow negative values of d; this is useful since we have taken each state at level 0 to be itself infinitely divisible, so that the tree is scale invariant under all dilatations. Transitions between microstates whose distances to the origin are $d \le \ell$ do not affect the "quasi-position" ℓ. The tree shown in the figure has branching ratio $\mu = 2$.

Comparing with (11) one sees that if one makes the identification $a^2 = i\Delta t$, then the canonical commutation relation becomes

$$[P, X] = -i\tilde{T}. \tag{14}$$

Note that this goes to the usual limit as Δt (or a) $\to 0$.

Going over to the transformed space, with Y straightforwardly implying multiplication by y, the following "q-deformed" commutation relations,

$$[\partial_y^{(q)}, Y]_q \equiv \partial_y^{(q)} Y - q Y \partial_y^{(q)} = 1 \tag{15}$$

and

$$[\overline{\partial}_y^{(q)}, Y]_{q^{-1}} = 1 \tag{16}$$

$$[\overline{\partial}_y^{(q)}, \partial_y^{(q)}]_q = 0, \tag{17}$$

hold. The transformed momentum and Hamiltonian operators remain Hermitian. However, they satisfy Heisenberg's equations of motion with the ordinary definition of the commutator and not with the deformed definition. To be able to give an intepretation of the physics, moreover, one has to transform back to the linear lattice.

We would now like to propose a different choice for the momentum operator. Notice that there is a kind of democracy between the right and left difference operators (3, 4), which makes it natural for the (self-adjoint) momentum operator on the discrete lattice to be defined [7] as in equation (9) but this democracy does not hold between $\partial_y^{(q)}$ and $\overline{\partial}_y^{(q)}$ which describe processes at *different scales*. On the linear chain, exactly one unit is added to an interval everytime a step is made to the right wherever one may be on the chain. However, when ℓ is increased by unity in y space, the size of the interval which is certain to include the origin increases by $(q-1)q^\ell$. We will therefore deliberately allow the momentum not to be an observable. This gives us the freedom to associate the momentum operator directly with the q-derivative (1)

$$P_q = -i\,\partial_y^{(q)}. \tag{18}$$

Now we consider the ordinary commutator of Y and P_q rather than the q-deformed one as in (15). We find that the canonical commutation relation becomes

$$[P_q, Y] = -i\,A_y^{(q)}. \tag{19}$$

To be consistent in our use of the difference operators, we write the deformed "Schrödinger equation" also in terms of the Jackson derivative, in this case with respect to time. Clearly, the dilatation factor for time, q_t, need not be equal to q; in fact we may define a "dynamical exponent" ζ *via* the relation

$$q_t = q^\zeta. \tag{20}$$

Defining the time dilation operator, T, by

$$Tf(y, t) \equiv f(y, q_t t) \tag{21}$$

we have $\partial_t^{(q_t)} \equiv (T-1)/[(q_t - 1)t]$. The Schrödinger equation then becomes

$$i\,\partial_t^{(q_t)} f(y, t) = H_q f(y, t), \tag{22}$$

where H_q is our — as yet unspecified — Hamiltonian. From (22), we have the relation

$$T = 1 - i(q_t - 1)t H_q, \tag{23}$$

to be compared with (13). We now need to make a choice either for H_q, or, equivalently, for T. In making this choice, we are guided by equation (14), and set

$$[P_q, Y] = -i\,T \tag{24}$$

yielding

$$T = A_y^{(q)}. \tag{25}$$

This is clearly a crucial step, which makes the motion of the phase point along the nonlinear chain ballistic, and in the increasing y direction. In the next section, we will discuss how this motion can be understood in terms of an associated process, that of a diffusing particle on an underlying hierarchical lattice, and therefore leads to nontrivial results.

Using (21, 22) one readily has

$$H_q = i\,\frac{T-1}{(q_t - 1)t}, \tag{26}$$

or, with (25, 7),

$$H_q = \frac{i(q-1)Y\partial_y^{(q)}}{(q_t - 1)t}. \tag{27}$$

The constant prefactor in equation (27) may be written as the inverse of a "basic number" [10],

$$[\zeta]_q \equiv \frac{q^\zeta - 1}{q - 1}, \tag{28}$$

where ζ is the dynamical exponent defined in (20). (This dynamical exponent ζ, which tells us how time scales with the distance, takes the value of 2 on Euclidean space; we expect it to be equal to the random walk dimension [11] on the hierarchical lattice.) With these definitions, the Hamiltonian operator becomes

$$H_q = -\frac{1}{[\zeta]_q}\frac{Y P_q}{t}, \tag{29}$$

which has the right "dimensions" for being an energy.

Note that this "Hamiltonian" is non-Hermitian, as is our "momentum," so that the energy is not an observable, neither is it a constant of the motion; H_q depends explicitly on time. Since $[H, P_q] \neq 0$, the momentum is not conserved either.

The solutions of the "Schrödinger equation" (22) can be found by making a separation of variables. Taking $f(y,t) = g(y)h(t)$ and using (29, 18), one has

$$\frac{t\partial_t^{(q_t)}h(t)}{h(t)} = \frac{1}{[\zeta]_q}\frac{Y\partial_y^{(q)}g(y)}{g(y)} . \tag{30}$$

Setting both sides of the equation equal to a constant, C, gives

$$\frac{t\partial_t^{(q_t)}h(t)}{h(t)} = C, \tag{31}$$

and

$$\frac{Y\partial_y^{(q)}g(y)}{g(y)} = [\zeta]_q C. \tag{32}$$

The solutions to these equations are given in terms of homogeneous functions, namely power laws, up to multiplication by oscillatory functions,

$$h(t) = F_{q_t}(t)t^{\psi}, \tag{33}$$

$$g(y) = F_q(y)y^{\chi}. \tag{34}$$

From (20), we find $[\zeta]_q[\psi]_{q_t} = [\zeta\psi]_q$. On the other hand, from (32, 34), we have $[\chi]_q = [\zeta]_q[\psi]_{q_t}$, whence, $\chi = \zeta\psi$. For finiteness as $t \to \infty$, χ, $\psi < 0$.

The oscillatory amplitudes multiplying the power laws in (33, 34) must satisfy $F_r(ru) = F_r(u)$ so that $\partial_u^{(r)}F_r(u) = 0$. Such functions periodic in the logarithm of their arguments can be expressed in terms of the Jackson integral [4,9] from 0 to ∞

$$F_r(u) = \int_{0u}^{\infty u} \phi(v)\mathrm{D}_v^{(r)} \tag{35}$$

$$\equiv \int_0^u \frac{\phi(v)}{v^{1+\omega}}\mathrm{D}_v^{(r)} + \int_u^{\infty} \frac{\phi(v)}{v^{1+\omega}}\mathrm{D}_v^{(r)} \tag{36}$$

$$= (r-1)u^{-\omega}\sum_{k=-\infty}^{\infty} r^{-k\omega}\phi(r^k u), \tag{37}$$

for $r > 1$, where we have used [2] the notation $\mathrm{D}_v^{(q)}$ for the q-differential of v; $\phi(v)$ is an arbitrary periodic function, which vanishes at the origin together with its first n_0 derivatives, n_0 being the smallest integer $> \omega$, and $\omega > 0$. Notice that since k ranges over all positive *and* negative values, r here may be bigger than unity, as we have assumed q to be.

Finally, the solutions of equation (22) can be written

$$f_{\psi}(y,t) = F_q(y)F_{q_t}(t)(y^{\zeta}t)^{\psi}. \tag{38}$$

These solutions are degenerate with respect to the functions ϕ and the indices ω appearing in the oscillatory amplitudes. Notice, however, that when y and t are only allowed to take discrete values such as q^m, q_t^n, with integer m and n, clearly the functions $F_q(y)$ and $F_{q_t}(t)$ can only take on constant values.

3 The "quasi-position" operator and spreading of the probability distribution

With the Hermitian operator Y we will associate a position-like observable which we will call the "quasi-position" [12], the "quantum numbers" ℓ corresponding to the highest level so far attained by the phase point on the y-lattice. To the motion of the phase point along the chain of quasi-positions q^{ℓ} (see Fig. 1a), there corresponds an associated process, that of the diffusive motion of a particle on an underlying hierarchical lattice as shown in Figure 1b.

To proceed from the ℓth level of the hierarchy to the next, we assume the particle has to surmount an energy barrier of hight R^{ℓ}, hence the hierarchy of characteristic times, q_t^{ℓ}. On this lattice, to each successive quasi-position indexed by the quantum number ℓ there corresponds a geometrical increase in the number of microstates [13,14]. The origin has been arbitrarily chosen at y_0. The regions of extent q^{ℓ} (or phase-space volume μ^{ℓ}, with μ being the branching ratio) over which the particle is successively delocalized form a nested hierarchy, *i.e.*, the microstates already available at the $\ell - 1$st level are subsumed by the microstates that become available at the ℓth level, with an increase in the 0-level states of $\mu^{\ell}(1 - \mu^{-1})$. The crucial point to realize is that *transitions between microstates within the same interval of size q^{ℓ} do not change the quantum number* ℓ, *i.e.*, the quasi-position. Going from one level to the next, however, means surmounting one higher barrier, therefore an increase in the phase space volume made available. Thus it implies an increase in the entropy of the system, and we expect this motion to be irreversible, which it indeed is [6].

It is useful to recall [15] that the quantum mechanical expectation of the transition of a free particle between two different space points on ordinary space can be associated with a weighted sum over all possible paths of a classically diffusing particle between these two points. The path integral over *quasi-positions*, however, is trivial — once the phase point has progressed to some level ℓ, the paths of the diffusing particle which go back and explore sites within the phase space volume already broached (at levels $\leq \ell$) simply do not contribute. Therefore the time dilation leads deterministically to an increase in ℓ, as can be seen from equation (25). On the other hand, the probability distribution for finding the particle at some level $\leq \ell$ at a given time t is *not* trivial, as will be described below.

The unnormalized state functions corresponding to the pure states $|\ell\rangle$ of the quasi-position operator are given, again up to multiplication by functions doubly periodic in $\ln y$ and $\ln t$ with periods $\ln q$ and $\ln q_t$, by

$$\epsilon_{\ell}(y,t) = \exp\left\{-\frac{1}{2}\left[\frac{(y/y_0)^{\zeta}}{\tau_{\ell}}t\right]^{\lambda}\right\} \tag{39}$$

where $\tau_{\ell} = R^{\ell}$ are the characteristic decay times, and $\lambda > 0$ is arbitrary. For simplicity, we shall choose $\lambda = 1$, but this does not at all affect the subsequent discussion. By (25), $\mathrm{T}\epsilon_{\ell}(y,t) = \mathrm{A}_y^{(q)}\epsilon_{\ell}(y,t)$. Thus, one must have

$$\epsilon_{\ell}(y, q_t t) = \epsilon_{\ell}(qy, t). \tag{40}$$

Substituting from (39), one finds that if $R \equiv q_t$, then we also have

$$A_y^{(q)} \epsilon_\ell(y,t) = \epsilon_{\ell-1}(y,t) \tag{41}$$

$$\overline{A}_y^{(q)} \epsilon_\ell(y,t) = \epsilon_{\ell+1}(y,t). \tag{42}$$

The expectation value of the quasi-position operator is to be computed using the definition of the scalar product [7],

$$\langle a,b \rangle_{y_0} \equiv \int_{0y_0}^{\infty y_0} \frac{a(y)\,b(y)}{y} D_y^{(q)} \tag{43}$$

$$\equiv (q-1) \sum_{k=-\infty}^{\infty} a(q^k y_0)\, b(q^k y_0). \tag{44}$$

Here y_0 serves as the origin of this hierarchical lattice, and could be chosen equal to unity.

Defining $\langle \epsilon_\ell(y,t), y\,\epsilon_\ell(y,t) \rangle_{y_0} \equiv Q_\ell(t)$, we have

$$Q_\ell(t) = (q-1)y_0 \sum_{k=-\infty}^{\infty} q^k e^{-q_t^{k-\ell} t}. \tag{45}$$

We see that the choice of the exponential form for the state functions is dictated by the requirement that they decay sufficiently fast for the infinite sum to converge at both ends. The meaning of $Q_\ell(t)$ is the probability, at time t, for the particle to be still found within some level $\leq \ell$. Notice that $Q_{\ell+1}(t) = Q_\ell(q_t^{-1}t)$ or $Q_\ell(t) = T Q_{\ell+1}(t)$. By (45) we have

$$Q_{\ell+1} = (q-1)y_0 \sum_{k=-\infty}^{\infty} q^k e^{-q_t^{k-1-\ell} t}. \tag{46}$$

Upon redefining the dummy index to be $k' = k - 1$, this gives

$$Q_{\ell+1}(t) = q\, Q_\ell(t). \tag{47}$$

Thus, clearly, $Q_\ell(t) = q^\ell Q_0(t)$ and the $\epsilon_\ell(y,t)$ span a representation of the algebra generated by the $\partial_y^{(q)}$, $\overline{\partial}_y^{(q)}$ and Y.

Now we would like to show that the kinetics imply that a probability distribution initially localized within an interval q^ℓ of the origin will spread in time in such a way that the uncertainty in the position becomes precisely as large as the whole phase space available at that energy. This means that the probability distribution is essentially uniform over the available phase space at any given time.

The absolute value of the uncertainty in the simultaneous determination of the "momentum" and "position" operators can be found as usual from the canonical commutation relation. In our case, from (19, 26) we have

$$|\langle \Delta Y \Delta P_q \rangle| \geq |\langle [Y, P_q] \rangle| \tag{48}$$

$$= |\langle i A_y^{(q)} \rangle| \tag{49}$$

$$= |\langle -(q-1)Y P_q + i \rangle|. \tag{50}$$

This tells us that the product of the uncertainty in the value of the position and the momentum operators is larger than the expectation value of their product in absolute value. Heuristically, one may say that, if $Y \sim v t^{1/\zeta}$ where v is some effective diffusivity, then the uncertainty $|\langle \Delta Y \Delta P_q \rangle| > |v t^{1/\zeta} p_q|$, where p_q is the average momentum for this Y eigenstate. Thus, the uncertainty in the position is as large, and increases with time in the same way, as the interval over which the particle or the phase point has travelled within the time t, i.e., it is equally likely to be found anywhere within the phase space volume it is energetically allowed to explore.

More precisely, the expectation value of $[Y, P_q]$, taken with respect to the solutions of the Schrödinger equation, normalized by their scalar product, yields

$$\langle [Y, P_q] \rangle = i \langle [(q-1)Y \partial_y^{(q)} + 1] \rangle \tag{51}$$

$$= i \left[(q-1)\frac{q^{\zeta\psi}-1}{q-1} + 1 \right] = i q^\chi. \tag{52}$$

With $q^\zeta = q_t$, this yields

$$|\langle \Delta Y \Delta P_q \rangle| \geq q^\chi = q_t^\psi. \tag{53}$$

On the other hand, taking the expectation value of the canonical commutator between the states $|\ell\rangle$ and using (24, 25), and (41, 42) gives us $\langle \epsilon_\ell, \epsilon_{\ell-1} \rangle$, which may be interpreted as a transition probability between the states $|\ell-1\rangle$ and $|\ell\rangle$. This is again consistent with the fact that uncertainty is a function of the leakage of the phase point to larger and larger regions of the phase space, as time goes by.

4 Discussion and connection with q-statistics

In this paper we have pointed out that on a lattice with equal spacing on the logarithmic scale, a natural choice for the position ("quasi-position") operator, together with a momentum operator that respects the asymmetry of the y-space, yields a kinetics that can be understood in terms of diffusion on an underlying ultrametric space, if one also identifies the canonical commutator with the time translation (here dilation) operator, as happens [7] on an ordinary periodic lattice. The motion to which this non-conventional kinetics corresponds is irreversible, with an explicit violation of time reversal symmetry resulting from the spreading with time of a probability distribution over a larger and larger volume of the phase space [6]. It might be noted that our Schrödinger equation (22, 27) involves, on the RHS, only the first derivative with respect to position, in accordance with the fact that diffusion on the hierarchical lattice corresponds to simply a drift with respect to the quasi-position. This makes the Schrödinger equation resemble the Fokker-Planck equation rather than the diffusion equation.

In statistical physics, hierarchical lattices have arisen recently in the anomalous relaxation of spin glasses [16,17], transport in random media [18] and fully developed turbulent media [13] as realizations of ultrametric spaces [20]. They consist of a hierarchy of nested intervals (see Fig. 1), and one may associate a geometrical

progression of spatial (and/or temporal) scales with the different levels of the hierarchy. Diffusion on ultrametric spaces have been thoroughly studied (see [13, 14, 16–20] and references therein) by other methods, including the renormalization group.

We would like to mention that Dimakis and Tzanakis [21] have also recently given an alternative description of the kinetics of open systems, built upon the assumption that observables are now defined on a manifold with non-commutative geometry. In this way, they recover the non-conventional calculus obeyed by stochastic differentiation (Itô's calculus), without making any uncontrolled approximations with respect to the microscopic Hamiltonian dynamics of the system. The relationship between our approaches, however, will be the subject of a different study.

Finally, we would like to make a connection with recent work on random sets and q-distributions. It has been remarked by Arık *et al.* [22] that the basic number $[n]_q$ with $q = 1 - 1/M < 1$ is the average number of distinct elements in a set which is contructed in n steps by making random draws from a source set with infinitely many elements of which there are M distinct kinds. In our case, $q > 1$, which is complementary to that considered by Arık *et al.* The spreading of the distribution in the phase space of our system extends at each step by $(1 - 1/q)q^m = (q - 1)q^{m-1}$, so that the total volume explored in n steps is precisely $1 + (q - 1)\sum_1^n q^{m-1} = q^n$.

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